Simulation Monte-Carlo

Exercice 3

Soit $U_1, U_2 \overset{\perp\!\!\!\perp}{\sim} \mathcal{U}_{[-1,1]}$ tq $U_1^2+U_2^2 \leq 1$

• Distribution de $U_1, U_2 \mid U_1^2+U_2^2 \leq 1$

$\rightarrow P(U_1^2+U_2^2 \leq 1) = P((U_1, U_2) \in B(0,1))$

boule euclidienne unité de $\mathbb{R}^2$

$= m(B(0,1)) / m([-1,1]^2)$, $m :=$ mesure de Lebesgue

$= \pi/4$

$\rightarrow$ Soit $A \in \mathcal{B}(\mathbb{R}^2)$,

$$P((U_1,U_2) \in A \mid U_1^2+U_2^2 \leq 1) = P(A \cap B(0,1))/P(U_1^2+U_2^2 \leq 1)$$

$$= \frac{m(A \cap B(0,1)) / m([-1,1]^2)}{\pi/4}$$

$$= m(A \cap B(0,1))/\pi$$

$\Rightarrow U_1, U_2 \mid U_1^2+U_2^2 \leq 1 \sim \mathcal{U}nif(B(0,1))$

• Changement de variable

Coordonnées polaires

$$\begin{cases} U_1 = \sqrt{S}\cos\theta \\ U_2 = \sqrt{S}\sin\theta \end{cases}, \quad \theta \in [0, 2\pi], \quad S = U_1^2+U_2^2$$

Soit $J = \begin{pmatrix} -\sqrt{S}\sin\theta & \cos\theta/2\sqrt{S} \\ \sqrt{S}\cos\theta & \sin\theta/2\sqrt{S} \end{pmatrix}$ (matrice « jacobienne »)

$$|\det(J)| = |-\sin^2\theta/2 - \cos^2\theta/2| = \frac{1}{2}$$

$$\Rightarrow f_{\theta,S}(\theta, s) = f_{U_1,U_2 \mid U_1^2+U_2^2 \leq 1}(\sqrt{s}\cos\theta, \sqrt{s}\sin\theta) \times \frac{1}{2}$$

donc $f_{\Theta,S}(\theta, s) = \mathbb{1}_{B(0,1)}(\sqrt{s}\cos\theta, \sqrt{s}\sin\theta) \times \frac{1}{\pi} \times \frac{1}{2}$

or $(\sqrt{s}\cos\theta, \sqrt{s}\sin\theta) \in B(0,1)$

$$\iff \begin{cases} \theta \in [0, 2\pi] \\ s \leq 1 \end{cases}$$

donc $f_{\Theta,S}(\theta, s) = \frac{1}{2\pi} \mathbb{1}_{[0,2\pi]}(\theta) \, \mathbb{1}_{[0,1]}(s)$

$\Rightarrow \quad S \sim U_{[0,1]} \perp\!\!\!\perp \Theta \sim U_{[0,2\pi]}$

(Rappel Box-Müller : $Z_1, Z_2 \sim U_{[0,1]}$
$X = \sqrt{-2\lg Z_1} \sin(\underbrace{2\pi Z_2}_{\sim U_{[0,2\pi]}})$, $Y = \sqrt{-2\lg Z_1} \cos(\underbrace{2\pi Z_2}_{\sim U_{[0,2\pi]}})$
$\Rightarrow X, Y \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(0,1)$)

OR $\tan\theta = \frac{U_2}{U_1} \Rightarrow \begin{cases} \cos\theta = \frac{U_1}{\sqrt{U_1^2+U_2^2}} = \frac{U_1}{\sqrt{S}} \\ \sin\theta = U_2/\sqrt{S} \end{cases}$

Soit $X = \sqrt{-2\lg S} \times U_2/\sqrt{S} = \sqrt{-2\lg S/S} \times U_2$
$Y = \sqrt{-2\lg S} \times U_1/\sqrt{S} = \sqrt{-2\lg S/S} \times U_1$

Alors $X, Y \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(0,1)$